Math 210

Quiz # 1, October 12, 2013

1. (a) Define each of the following: limit point, open set, compact set, bounded set.

(b) Prove that a closed subset of a compact set is compact.

(c) If $E \subset K$ and K is compact, prove that $\overline{E} \subset K$.

2. State without proof the completeness property of $\mathbb{R}$, the Archimedean property of $\mathbb{R}$, the Heine-Borel Theorem, and the Bolzano-Weierstrass Theorem (for sequences).

3. (a) Use the completeness property of $\mathbb{R}$ to prove that every monotone decreasing sequence which is bounded below is convergent.

(b) Let c_n be the sequence defined by

$$c_n = \frac{2}{1+1^2} + \frac{2.2}{1+2^2} + \cdots + \frac{2n}{1+n^2} - \ln(1+n^2).$$

Prove that c_n is a convergent sequence.

4. A sequence a_n is defined by

$$a_{n+1} = a_n + \frac{n}{(1+n^2)a_n}, a_1 = 1.$$

Obtain the asymptotic behaviour of a_n and prove your answer.

5. Given that $\sum_{k=1}^{\infty} b_k$ is a convergent series of positive terms. Suppose that $\{a_k\}$ is a sequence satisfying the inequality

$$|a_{k+1} - a_k| \leq c_k, k \in \mathbb{N}.$$

Prove that a_k is a convergent sequence.

6. Let $\sum_{k=1}^{\infty} a_k$ be a series of positive terms, and use the notation

$$A_n = \sum_{k=1}^{n} a_k, A = \sum_{k=1}^{\infty} a_k.$$

Suppose that

$$\limsup_{k\to\infty} \frac{a_{k+1}}{a_k} = \alpha,$$

that $\alpha < 1$, and take r such that $\alpha < r < 1$.

(a) Prove that the series is convergent.

(b) Prove that there exists $N > 0$ such that

$$0 \leq A - A_n \leq \frac{a_N r^{n-N+1}}{1-r}, n > N.$$